A Gerhardian Puzzle Book

October 7, 2022

Foreword

Dear reader,

you hold in your hands a short collection of math puzzles created by Gerhard, a craft in which he took great pleasure. One of these puzzles was taken from a legendary university exam, the others were proposals for math competitions such as the Austrian Mathematics Olympiad (ÖMO), the International Mathematics Olympiad (IMO), and the Mediterranean Mathematics Competition (MMC).

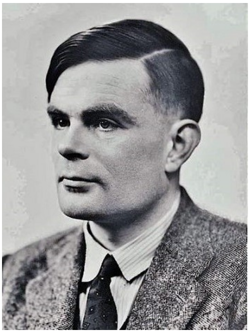
We hope this booklet is a nice memory to those who have known Gerhard.

The editors
October 2022

Puzzles

Puzzle #1: The Hall of Fame

Name the following three superstars of theoretical computer science:



Puzzle #2: Problem Submission for IMO 2021

Let $S_1, \ldots, S_m$ be distinct subsets of $\{1, 2, \ldots, 2021\}$ such that the intersection of any two sets contains a triple of consecutive integers (that is, integers $x, x+1, x+2$ for some $x \geq 1$). Determine the largest possible value of m.

Puzzle #3: The Ultimate Sudoku

Determine the largest integer N, for which there exists an $N \times 100$ table T that has the following properties:

(i) Every row contains the numbers $1, 2, \ldots, 100$ in some ordering.

(ii) For any two rows $r \neq s$, there exists a columns c such that $|T(r, c) - T(s, c)| \geq 2$.

Puzzle #4: A Pandemic of the Rationals

The Corona virus is attacking the positive rational numbers. All rational numbers $x > 1$ have already been infected. The virus has only two ways to spread:

- Whenever a rational number x is infected, the virus also spreads to $\frac{x}{2x+1}$.
- Whenever a rational number x is infected, the virus also spreads to every integer multiple kx of x with $k \geq 2$.

Determine all positive rational numbers that are safe from infection by the virus.

Puzzle #5: Find a Name for this Puzzle

Let $a_0, a_1, \ldots$ be an infinite sequence of integers with $a_0 = a_1 = 0$ that satisfies the following properties:

(i) For every $n \geq 2$, there exist three integers p, q, r with $0 \leq p, q, r \leq n-1$ and $p + q + r = n$ such that $a_n = a_p + a_q + a_r + 24pqr$.

(ii) There exist infinitely many integers n for which $a_n \geq n^3 - n$.

Prove that this sequence contains infinitely many multiples of 2021.

Puzzle #6: On the Spread of Numbers

For a sequence $a_1 \leq a_2 \leq \ldots \leq a_n$ of reals, define n^2 non-negative reals $b_{i,j} = |a_i - a_j|$ with $1 \leq i, j \leq n$. Net N_1 denote the number of $b_{i,j}$s that satisfy $b_{i,j} \leq 1$, and let N_3 denote the number of $b_{i,j}$s that satisfy $b_{i,j} \leq 3$. Prove that $N_3 \leq 5N_1$.

Puzzle #7: Primes of a Polynomial

Determine all integers $n \geq 1$ for which the number $n^8 + n^6 + n^4 + 4$ is prime.

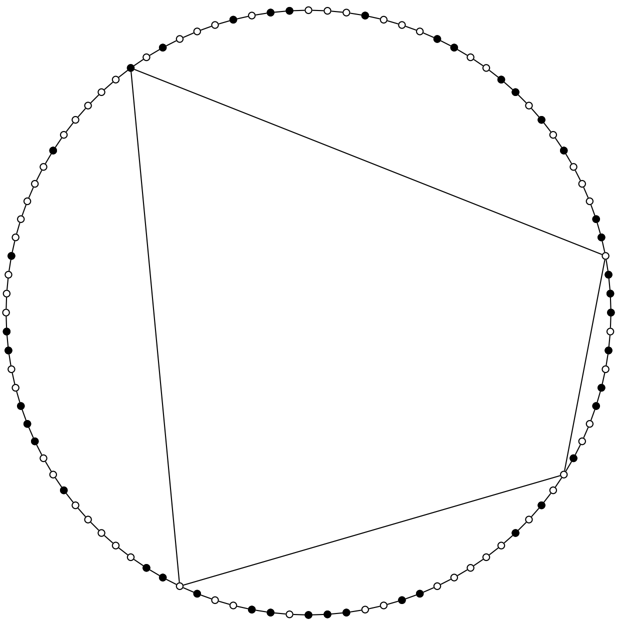
Puzzle #8: Radio Beams

The point set P consists of all points (x, y) with integer coordinates $1 \leq x, y \leq 99$. A radio signal originating from a point (a, b) is received by all the points (x, b) in the same row and by all the points (a, y) in the same column. Find a smallest set $R \subseteq P$ of radio stations, such that no two distinct points in P receive signals from precisely the same stations in R.

Puzzle #9: Quadrangling the 100-gon

In a regular 100-gon, 41 vertices are colored black and the remaining 59 vertices are colored white. Prove that there exist 24 quadrangles $Q_1, \ldots, Q_{24}$ whose corners are vertices of the 100-gon, so that

- distinct quadrangles Q_i and Q_j are always disjoint, and
- every quadrangle Q_i has three corners of one color and one corner of the other color.



Puzzle #10: A Cookie for a Number

Players A and B play a game with a blackboard that initially contains 111 copies of the number 1. In every round, player A erases two numbers x and y from the blackboard, and then player B writes one of the numbers $x+y$ and $|x-y|$ on the blackboard. The game terminates as soon as one of the numbers on the blackboard is larger than the sum of all other numbers. Player B must then give as many cookies to player A as there are numbers on the blackboard.

Player A wants to get as many cookies as possible, whereas player B wants to give as few as possible. Determine the number of cookies that A receives if both players play optimally.

Puzzle #11: Overlapping Integer Sequences

Two infinite integer sequences are given by $a_0 = b_0 = 2$ and $a_1 = b_1 = 14$, and for $n \geq 2$ by

$$a_n = 14a_{n-1} + a_{n-2},$$
$$b_n = 6b_{n-1} - b_{n-2}.$$

Decide whether there exist infinitely many integers that occur in both sequences.

Puzzle #12: Integer Sequences

Determine the smallest integer n, for which there exist integers $x_1, \ldots, x_n$ and positive integers $a_1, \ldots, a_n$ so that

$$x_1 + \cdots + x_n = 0, \qquad a_1x_1 + \cdots + a_nx_n > 0, \qquad a_1^2x_1 + \cdots + a_n^2x_n < 0.$$

Puzzle #13: Perfect Squares

For $n \geq 1$, let $a_n = n^4 + 3n^2 + 4$, and let $b_n = a_1a_2 \ldots a_n$. Decide whether there exists an index $n \geq 2017$ for which b_n is a perfect square.

(Note: A perfect square is the square of an integer.)

Puzzle #14: Balearic Sets

A set S of integers is *Balearic*, if there are two (not necessarily distinct) elements $s, s' \in S$ whose sum $s + s'$ is a power of two; otherwise it is called a *non-Balearic* set. Find an integer n such that $\{1, 2, \ldots, n\}$ contains a 99-element non-Balearic set, whereas all the 100-element subsets are Balearic.

Puzzle #15: Recursive Bounds

Consider a function $f : \mathbb{N} \to \mathbb{R}$ defined by $f(1) = \frac{3}{2}$ and $f(n)f(n+1) = f(n) + n$ for $n \geq 1$. Determine an integer m so that $\lfloor f(m) \rfloor = 2018$. (Note: $\lfloor x \rfloor$ denotes the largest integer $\leq x$.)

Puzzle #16: The Tiny Sudoku

Determine the largest integer N, for which there exists a $6 \times N$ table T that has the following properties:

(i) Every column contains the numbers $1, 2, \ldots, 6$ in some ordering.

(ii) For any two columns $i \neq j$, there exists a row r such that $T(r, i) = T(r, j)$.

(iii) For any two columns $i \neq j$, there exists a row s such that $T(s, i) \neq T(s, j)$.

Puzzle #17: The Mycenaean Revenge

An integer $a \geq 1$ is called *Aegean*, if none of the numbers $a^{n+2} + 3a^n + 1$ with $n \geq 1$ is prime. Prove that there are at least 500 Aegean integers in the set $\{1, 2, \ldots, 2018\}$.

Puzzle #18: Composites Revisited

Let S be a set of $n \geq 2$ positive integers. Prove that there exist at least n^2 integers that can be written in the form $x + yz$ with $x, y, z \in S$.

Puzzle #19: An Enlightening Inequality

Let $m_1 < m_2 < \cdots < m_s$ be a sequence of $s \geq 2$ positive integers, none of which can be written as the sum of (two or more) distinct other numbers in the sequence. For every integer r with $1 \leq r < s$, prove that

$$r \cdot m_r + m_s \geq (r+1)(s-1).$$

Puzzle #20: Small Digit Sums

Prove that there exist infinitely many positive integers x, y, z for which the sum of the digits in the decimal representation of $4x^4+y^4-z^2+4xyz$ is at most 2.

Puzzle #21: Euclid in China

Determine all integers $m \geq 2$ for which there exists an integer $n \geq 1$ with $\gcd(m, n) = 1$ and $\gcd(m, 4n + 1) = 1$.

Puzzle #22: Yet Another One With Numbers

Let A be a set of (positive or negative) integers, and let $m \geq 2$ be an integer. Assume that there exist non-empty subsets $B_1, B_2, B_3, \ldots, B_m$ of A whose elements add up to the sums $m^1, m^2, m^3, \ldots, m^m$, respectively. Prove that $|A| \geq \frac{m}{2}$.

Puzzle #23: Prime Fractions

For every sequence $p_1 < p_2 < p_3 < p_4 < p_5$ of prime numbers, determine the largest integer N for which the following equation has no solution in positive integers $x_1, \ldots, x_5$:

$$\frac{x_1}{p_1} + \frac{x_2}{p_2} + \frac{x_3}{p_3} + \frac{x_4}{p_4} + \frac{x_5}{p_5} = \frac{N}{p_1 p_2 p_3 p_4 p_5}$$

Puzzle #24: The Glastonbury Festival Problem

Three positive integers p, q, r form a *Somerset-triple* (p, q, r), if p and q are primes and if $pq - r \neq 0$ is divisible by $p - r$ and by $q - r$.

a. Prove that there exist infinitely many Somerset-triples.

b. Prove that there exist only finitely many Somerset-triples (p, q, r) with $r \leq 1.000.000$.

Puzzle #25: Infinite Pairs

Decide whether there exist infinitely many pairs (a, b) of positive integers for which $a^2 - a + b^3 + 1$ is a multiple of ab.

Puzzle #26: Where is the Fence?

Determine the largest integer n, for which there exists a set S of n points in $\mathbb{R}^3$ with the following property: For every subset $T \subseteq S$, there exists an axes-parallel cube that has all points of T in its interior or on its boundary, and that has all points of $S \setminus T$ in its exterior.

Solutions

Solution for Puzzle #1

The persons in the pictures are Alan Turing, David Hilbert, and Kurt Gödel – listed from left to right.

Solution for Puzzle #2

The largest possible value is $m = 2^{2018}$. This value can be reached for instance by using all the subsets that contain the three numbers 1, 2, 3.

To see that no larger value is possible, we argue as follows. For $i = 1, \ldots, m$ define set A_i as the intersection of S_i with the set of integers $\equiv 0 \mod 3$. Hence every A_i is a subset of the 673-element set $\{3, 6, 9, 12, \ldots, 2019\}$. If some A_i is the complement of some A_j, then the intersection $S_i \cap S_j$ will only contain numbers $\equiv 1, 2 \mod 3$ and hence cannot contain a triple of consecutive integers; a contradiction. Therefore there are at most 2^{672} distinct sets A_i.

Analogously, we define B_i as the intersection of S_i with the set of integers $\equiv 1 \mod 3$, and C_i as the intersection of S_i with the set of integers $\equiv 2 \mod 3$. Analogous arguments show that there are at most 2^{673} distinct sets B_i and at most 2^{673} distinct sets C_i. As the sets S_i are pairwise distinct, there are at most $2^{672} \cdot 2^{673} \cdot 2^{673} = 2^{2018}$ possibilities of taking the union of a set A_i, a set B_j, and a set C_k. This yields $m \le 2^{2018}$.

Solution for Puzzle #3

We show that the largest such integer is $N = \frac{100!}{2^{50}}$.

Non-existence of a larger table. Let us consider some fixed row in the table, and let us replace (for $k = 1, \ldots, 50$) the two numbers $2k - 1$ and $2k$ respectively by the symbol x_k. The resulting *pattern* is an arrangement of the 50 symbols $x_1, x_2, \ldots, x_{50}$, where every symbol occurs exactly twice Note that there are $N = \frac{100!}{2^{50}}$ distinct patterns $P_1, \ldots, P_N$.

If rows $r \neq s$ in the table have the same pattern P_i, then $|T(r, c) - T(s, c)| \le 1$ holds for all columns c. As this violates property (ii) in the problem statement, different rows have different patterns. Hence there are at most $N = \frac{100!}{2^{50}}$ rows.

Existence of a table with N rows. We construct the table by translating every pattern P_i into a corresponding row with the numbers $1, 2, \ldots, 100$ The translation goes through steps $k = 1, \ldots, 50$ and replaces the two occurences of sybol x_k by $2k-1$ and $2k$.

- The left occurence of x_1 is replaced by 1, and its right occurence is replaced by 2.
- For $k \geq 2$, we make the three numbers $2k-2, 2k-1, 2k$ show up (ordered from left to right) either in the order $2k-2, 2k-1, 2k$, or as $2k, 2k-2, 2k-1$, or as $2k-1, 2k, 2k-2$. This is possible, since the number $2k-2$ has been placed in the preceeding step, and shows up before / between / after the two occurences of the symbol x_k.

We claim that N rows that result from the N patterns yield a table with the desired property (ii). Indeed, consider the r-th and the s-th row ($r \neq s$), which by construction result from patterns P_r and P_s. Call a symbol x_i *aligned*, if it occurs in the same two columns in P_r and in P_s. Let k be the largest index, for which symbol x_k is not aligned. Note that $k \geq 2$. Consider the column c' with $T(r, c') = 2k$ and the column c'' with $T(s, c'') = 2k$. Then $T(r, c'') \leq 2k$ and $T(s, c') \leq 2k$, as all the symbols x_i with $i \geq k+1$ are aligned.

- If $T(r, c'') \leq 2k-2$, then $|T(r, c'') - T(s, c'')| \geq 2$ as desired.
- If $T(s, c') \leq 2k-2$, then $|T(r, c') - T(s, c')| \geq 2$ as desired.
- If $T(r, c'') = 2k-1$ and $T(s, c') = 2k-1$, then symbol x_i is aligned; contradiction.

In the only remaining case, we have $c' = c''$, so that $T(r, c') = T(s, c') = 2k$ holds. Now let us consider the columns d' and d'' with $T(r, d') = 2k-1$ and $T(s, d'') = 2k-1$. Then $d' \neq d''$ (as symbol x_k is not aligned), and $T(r, d'') \leq 2k-2$ and $T(s, d') \leq 2k-2$ (as all symbols x_i with $i \geq k+1$ are aligned).

- If $T(r, d'') \leq 2k-3$, then $|T(r, d'') - T(s, d'')| \geq 2$ as desired.
- If $T(r, c') \leq 2k-3$, then $|T(r, d') - T(s, d')| \geq 2$ as desired.

In the only remaining, case we have $T(r, d'') = 2k-2$ and $T(s, d') = 2k-2$. Now row r has the numbers $2k-2, 2k-1, 2k$ in the three columns d'', d', c', whereas row s has these numbers in the three columns d', d'', c'. As one of these triples violates the ordering property of $2k-2, 2k-1, 2k$, we have the final contradiction.

Solution for Puzzle #4

We claim that exactly the unit fractions (that is, the reciprocals $\frac{1}{n}$ of integers $n \geq 1$) are safe from infection. We start with some definitions and observations.

- We introduce the function $f(x) = \frac{x}{2x+1}$. Note that positive rational numbers $x < y$ always satisfy $f(x) < f(y)$.
- Whenever all rational numbers in some (open) interval with endpoints ℓ and r are infected, then also all rational numbers in the interval with endpoints $f(\ell)$ and $f(r)$ will be infected.

(1) For a non-negative integer $j \geq 0$ define the open interval $I_j = (\frac{1}{2j+1}, \frac{1}{2j})$. We prove by induction on $j \geq 0$ that all rational numbers $x \in I_j$ eventually will become infected. The anchor step with $j = 0$ concerns the interval $I_0 = (1, \infty)$, and hence follows directly from the problem statement. In the inductive step, we deduce from the infection of interval $I_{j-1} = (\frac{1}{2j-1}, \frac{1}{2j-2})$ with endpoints $\ell = \frac{1}{2j-1}$ and $r = \frac{1}{2j-2}$, that also all rational numbers in the open interval with endpoints $d(\ell) = \frac{1}{2j+1}$ and $f(r) = \frac{1}{2j}$ will become infected, and this is exactly the interval I_j.

(2) Next, consider an arbitrary rational number $x = \frac{a}{b}$ with $\gcd(a, b) = 1$ and $a \geq 2$. Note that all positive rational numbers except the unit fractions are of this form. The argument branches into two subcases. In the first subcase, assume that the denominator b is odd. Since the integers $2a$ and b are relatively prime, there exists a positive integer c so that $bc \equiv 1 \mod 2a$. In other words, there are positive integers c and j with $bc = 1 + 2aj$. Then $2aj < bc$ implies $\frac{a}{b} < \frac{c}{2j}$, and $bc < a + 2aj$ implies $\frac{a}{b} > c\frac{c}{2j+1}$, which may be summarized as

$$\frac{c}{2j+1} < x < \frac{c}{2j}.$$

This means that $\frac{x}{c} \in I_j$. Since $\frac{x}{c}$ becomes infected by (1), also x will become infected.

In the second subcase, assume that the denominator b is even with $b = 2b'$; note that in this case the (odd) numerator satisfies $a \geq 3$. Since the integers a and b' are relatively prime, there exists a positive integer c so that $b'c \equiv 1 \mod a$. In other words, there are positive integers c and j with $b'c = 1 + aj$. Then $aj < b'c$ implies $\frac{a}{b} < \frac{c}{2j}$, and $2b'c = 2 + 2aj < a + 2aj$ implies $\frac{a}{b} > \frac{c}{2j+1}$, so that again

$$\frac{c}{2j+1} < x < \frac{c}{2j}.$$

As in the first subcase, we conclude that $\frac{x}{c} \in I_j$ and that the virus will spread to x.

(3) It remains to show that unit fractions are safe from infection by this virus. For the sake of contradiction, we assume that this is not the case. Then consider the first unit fraction $\frac{1}{n}$ that becomes infected by the virus. If the infection has spread from some infected x to $f(x) = \frac{1}{n}$, then another unit fraction $x = \frac{1}{n-2}$ had been infected before; that's a contradiction. If the infection has spread from some infected x to $kx = \frac{1}{n}$ with $k \geq 2$, then another unit fraction $x = \frac{1}{kn}$ had been infected before; that's the final contradiction.

Solution for Puzzle #5

In the first step, we prove the statement (A) $a_n \leq n^3 - n$ for all $n \geq 0$ by induction:

The statement trivially holds true for $n = 0$ and $n = 1$. In the inductive step, we consider the three integers p, q, r from the problem statement (i) to get

$$\begin{aligned}
a_n &= a_p + a_q + a_r \\
&\leq (p^3 - p) + (q^3 - q) + (r^3 - r) + 24pqr \\
&= (p+q+r)^3 - (p+q+r) \\
&\qquad - 3(p^2q + p^2r + q^2p + q^2r + r^2p + r^2q - 6pqr) \\
&\leq n^3 - n.
\end{aligned}$$

Here the equality in the first line follows from (i). The inequality in the second line follows by applying the inductive assumption to p, q, r. The final inequality follows by applying the arithemtic-geometric mean inequality to the six numbers $p^2q, p^2r, q^2p, q^2r, r^2p, r^2q$.

In the second step, we prove by induction on $n \geq 1$ the statement (B): If $a_n = n^3 - n$, then (B1) $n = 3^k$ (with $k \geq 0$) is a power of 3, and (B2) $a_m = m^3 - m$ holds true for all smaller such powers $m = 3^\ell$ with $0 \leq \ell \leq k-1$. Indeed, statement (B) holds for $n = 1$ (where $k = 0$ and $a_1 = 0$). In the inductive step, we investigate the equality cases in the above chain of inequalities in the proof of (A). The equality case $a_n = n^3 - n$ enforces $a_p = p^3 - p$, $a_q = q^3 - q$ and $a_r = r^3 - r$ in the second line. Furthermore, we must have equality in the arithmetic-geometric mean inequality, which in turn implies $p = q = r$. The inductive assumption for (B) implies that $p = q = r = 3^{k-1}$ and that $n = p + q + r = 3^k$. Furthermore, $a_m = m^3 - m$ holds true for all $m = 3^\ell$ with $0 \leq \ell \leq k-1$. this completes the proof of (B).

Finally, as there exist infinitely many integers n for which $a_n \geq n^3 - n$, we conclude from (A) and (B) that $a_n = n^3 - n$ holds whenever $n = 3^k$ is a power of 3. By Euler's Theorem, there exist infinitely many k for which $3k \equiv 1 \mod 2021$. Hence, there are infinitely many $n = 3^k$ for which $a_n = (3^k - 1)3^k(3^k + 1)$ is a multiple of 2021. This completes the proof.

Solution for Puzzle #6

The proof is done by induction on n. Since $N_1 \geq n$ follows from $b_{i,i} \equiv 0$ and since $N_3 \leq n^2$ trivially holds, we get the starting point of the induction for $n \leq 5$.

In the inductive step to some $n \geq 6$, define $M_i := |\{j \mid a_i - 1 \leq a_j \leq a_i + 1\}|$ for $i = 1, \ldots, n$. let M^* denote the maximum of all M_i and let k be the smallest index with $M_k = M^*$. We partition the closed interval $[a_k - 3, a_k + 3]$ into five sub-intervals:

- The half open sub-interval $[a_k - 3a_k - 2)$ contains at most $M^* - 1$ of the a_js, as otherwise we would have a contradiction to the definition of k.

- Similarly, the half-open sub-interval $[a_k - 2, a_k - 1)$ contains at most $M^* - 1$ of the a_js.

- By definition, the closed interval $[a_k - 1, a_k + 1]$ contains exactly M^* of the a_js.
- The half-open sub-interval $(a_k + 1, a_k + 2]$ contains a most M^* of the a_js.
- The half-open sub-interval $(a_k + 2, a_k + 3]$ contains a most M^* of the a_js.

We apply the inductive assumption to the sequence of $n - 1$ reals that results by removing a_k from sequence $a_1, \ldots, a_n$; this yields the inequality $N_3' \leq 5N_1'$ for the corresponding numbers N_1' and N_3'. As the contribution of a_k to N_1 equals $2M^* - 1$, we get

$$N_1 = N_1' + 2M^* - 1.$$

As the contribution of a_k to N_3 is at most $2(M^* - 1) + 2(M^* - 1) + 2(M^* - 1) + 2M^* + 2M^*$, we get

$$N_3 \leq N_3' + 10M^* - 5.$$

The two displayed equations together with $N_3' \leq N_1'$ yield the desired inequality $N_3 \leq 5N_1$. This completes the inductive step, and also the inductive proof.

Solution for Puzzle #7

We use the factorization

$$n^8 + n^6 + n^4 + 4 = (n^4 - n^3 + n^2 - 2n + 2)(n^4 + n^3 + n^2 + 2n + 2).$$

The first factor $f(n)$ satisfies

$$f(n) = n^4 - n^3 + n^2 - 2n + 2 = n^3(n - 1) + (n - 1)^2 + 1,$$

and hence satisfies $f(n) \geq 2$ for all $n \geq 2$. The second factor $g(n) = n^4 + n^3 + n^2 + 2n + 2$ is strictly greater than 2 for all $n \geq 2$. This only leaves the case $n = 1$ as a potential candidate for a prime, and indeed $f(1)g(1) = 1 \cdot 7 = 7$ is prime.

Solution for Puzzle #8

Consider the set R with 148 stations in (x,x) for $x=1,\dots,99$ and in $(x,100-x)$ for $x=1,\dots,49$. It is easily verified that no two distinct points in P receive signals from the same subset of stations in R. We will show that there is no solution with $|R|<148$ stations.

For a set $R\subseteq P$ of radio stations and a point $p\in P$, let $R(p)$ denote the set of stations in the same row or column as p. Two points $p_1,p_2\in P$ with $p_1\neq p_2$ *collide* on R if $R(p_1)=R(p_2)$. We claim that any set R without colliding points contains at least 148 stations.

First assume that there is a row r without station. If there is another row $r'\neq r$ that contains no station, then the points $(r,1)$ and $(r',1)$ collide. If there is another row $r'\neq r$ that contains only a single station (r',y), then the points (r,y) and (r',y) collide. If each of the rows $r'\neq r$ contains at least two stations, we have the desired $|R|\geq 2\cdot 98\geq 148$.

It remains to consider the case where every row and (by symmetry) every column contain at least one station. We say that a row or column is *bad* if it contains a single station, and that it is *good* if it contains at least two stations. Consider a bad row r whose station is in (r,y); then we assign column y as partner to r. Symmetrically, a bad column c with station (x,c) gets row x as partner.

Lemma 1. *At most one bad row is matched with a bad column partner.*

Proof. If a bad row r is matched with a bad column c and another bad row r' is matched with another bad column c', then the points (r,c') and (r',c) would collide; contradiction. □

Lemma 2. *No good column is made partner of two distinct bad rows.*

Proof. If two bad rows r and r' both get the same good column c as partner, then the points (r,c) and (r',c) would collide; contradiction. □

Now let b denote the number of bad rows and columns, and let g denote the number of good rows and columns. Clearly $b+g=198$. If no bad row/column is matched with a bad column/row, then the injectivity of the partners yields $b\leq g$. If one bad row/column is matched with a bad column/row, then the injectivity of the partners yields $b-2\leq g$.

In either case we have $b - 2 \leq g$. The total number of radio stations satisfies

$$\begin{aligned} |R| &\geq \frac{1}{2}(b + 2g) &&= \frac{1}{2}(b + \tfrac{3}{2}g + \tfrac{1}{2}g) \\ &\geq \frac{1}{2}(b + \tfrac{3}{2}g + \tfrac{1}{2}(b-2)) &&= \frac{3}{4}(b+g) - \frac{1}{2} \cdot 2 \\ &= \frac{3}{4} \cdot 198 - 1 &&= 147.5. \end{aligned}$$

This gives the desired bound $|R| \geq 148$, and completes the argument.

Solution for Puzzle #9

Call a quadrangle *skew-colored*, if it has three corners of one color and one corner of the other color. In a first phase, we repeatedly find four consecutive points along the convex hull that form a skew-colored quadrangle and remove these four points from the picture. Assume that after some time we are left with b black and w white points, where

$$b + w = 4k \text{ and } k \leq b, w \leq 3k \text{ and } b \equiv w \equiv k \mod 2. \qquad (*)$$

We claim that if $b < w$, then we can find four consecutive points, so that three are white and one is black. Divide the points into groups of size 4 along the hull. Since $b < w$, one of these groups must contain more white points than black points. If this group consist of one black and three white points, we are done. If this group consist of four white points, consider the maximal sequence of white points along the hull that contains the group. The last three (white) points together with the subsequent black point then form the desired skew-colored quadrangle (note: the black point is guaranteed to exist, as $b \geq k$). After removing the skew-colored quadrangle we are left with $b' = b - 1$ black and $w' = w - 3$ white points, where $b' + w' = 4(k-1)$, and $k - 1 \leq b', w' \leq 3(k-1)$, and $b' \equiv w' \equiv k - 1 \mod 2$; hence (*) is satisfied again. if $b > w$, we proceed symmetrically to the case $b < w$.

In the second phase, we consider the case $b = w$. Let $p_1, \ldots, p_{4k}$ denote the points in clockwise order along the convex hull. If some four consecutive points form a skew-colored quadrangle, we remove them and maintain (*). Otherwise we argue as follows.

(1) Assume that two consecutive points p_1 and p_2 have the same color (say white). If p_3 is also white, we easily find four consecutive

points that form a skew-colored quadrangle; hence p_3 is black. If p_4 is white, then $p_1, \ldots, p_4$ form a skew-colored quadrangle; hence p_4 is black. Now p_3 and p_4 have the same color black, and we can show inductively that black and white points alternate along the convex hull.

For $i = 1, \ldots, k-1$ we form the skew-colored quadrangle p_{3i-2}, p_{3i-1}, p_{3i}, p_{4k-i-1}. The four points p_{3k-2}, p_{3k-1} and p_{4k-1}, p_{4k} remain unused.

(2) Finally assume that no two consecutive points have the same color. Then white and black points alternate along the convex hull.

For $i = 1, \ldots, k-1$ form the quadrangle $p_{3i-2}, p_{3i-1}, p_{3i}, p_{4k-i}$. The four points $p_{3k-2}, p_{3k-1}, p_{3k}$ and p_{4k} remain unused.

Since $b = 41, w = 59$ and $k = 25$ satisfy (*), the proof is complete.

Solution for Puzzle #10

If both players play optimally, player A will receive exactly *six* cookies from B.

A good strategy for player *A* For an integer x on the blackboard, let us define its *historical size* $H(x)$ as the total number of 1s from the initial situation that have been used to build x. Formally, for every integer $x = 1$ in the initial situation we set $H(x) = 1$. Whenever a new integer $z = x + y$ or $z = |x - y|$ is written on the board, we let $H(z) = H(x) + H(y)$. Consider the following strategy for player A: *"If the blackboard contains two copies of the same positive integer, then A selects two such copies as x and y.“* Under this strategy, every integer x on the blackboard will either be a zero or a power of two, and its historical size $H(x)$ will always be a power of two. Once all the non-zero integers on the blackboard are pairwise distinct, the game has terminated (since the largest power of two on the board is larger than the sum of all the remaining powers of two on the board). At any moment in time (and in particular at termination), the historical sizes of the blackboard numbers add up to 111. Note that for writing 111 as a sum of powers of two, we need at least 6 summands:

$$111 = 64 + 32 + 8 + 4 + 2 + 1.$$

Since at any moment in time the blackboard contains at least 6 numbers, the strategy indeed guarantees at least 6 cookies for player A.

A good strategy for player B At any moment in time, we designate one integer x on the blackboard as *leader* and associate a corresponding label $\ell(x) \in \{-1,+1\}$ with it. For the initial situation, we arbitrarily pick one of the integers $x = 1$ as leader and set $\ell(x) = 1$. Whenever the leader x (together with some other integer y) is erased, the newly written integer z becomes the new leader. If $z = x + y$ or $z = x - y$ then the new label is $\ell(z) = \ell(x)$; if $z = y - x$ then $\ell(z) = -\ell(x)$. In other words, the label is tracking the sign of the leader.

Next, assume that the blackboard contains the integers $a_1, \ldots, a_n$ with leader a_q. A system $c(a_1), \ldots, c(a_n)$ of coefficients $c(a_i) \in \{-1,+1\}$ is called *good*, if $c(a_q) = \ell(a_q)$ and if

$$c(a_1) \cdot a_1 + c(a_2) \cdot a_2 + \ldots + c(a_n) \cdot a_n > 0.$$

We denote by $g(\mathcal{C})$ the overall number of good coefficient systems for that blackboard contents $\mathcal{C}$. our next goal is to prove three lemmas on these values $g(\mathcal{C})$.

Lemma 1. *Let $\mathcal{C}$ be a blackboard contents. Assume that player A picks integers x and y with $x \geq y$, so that player B either generates a new scenacrio $\mathcal{C}_1$ by writing down $x + y$ or a new scenario $\mathcal{C}_2$ by writing down $x - y$. Then $g(\mathcal{C}) = g(\mathcal{C}_1) + g(\mathcal{C}_2)$.*

Proof. In a good coefficient system for $\mathcal{C}$, we replace the two terms $c(x)x$ and $c(y)y$ by the single term $c(z)z$ with $c(z) = c(x)$ and $z = x + c(x)c(y)y$. If x is the leader for $\mathcal{C}$, then $\ell(z) = c(x) = \ell(x)$ for $\mathcal{C}_1$ and $\mathcal{C}_2$. If y is the leader for $\mathcal{C}$, then for $\mathcal{C}_1$ we have $\ell(z) = c(x) = c(y) = \ell(y)$, and for $\mathcal{C}_2$ we have $\ell(z) = c(x) = -c(y) = -\ell(y)$. Hence, the resulting coefficient system is good for $\mathcal{C}_1$ or good for $\mathcal{C}_2$. This yields a bijection between the good systems for $\mathcal{C}$ and the good systems for $\mathcal{C}_1$ and $\mathcal{C}_2$, as one can uniquely recover $c(x)$ and $c(y)$ from z and $c(z)$. □

Lemma 2. *In the initial situation $\mathcal{C}_0$, the value $g(\mathcal{C}_0)$ is not divisible by 32.*

Proof. A coefficient system for $\mathcal{C}_0$ is good, if and only if the leader and at least 55 of the remaining 110 numbers all have coefficient $+1$. This yields $g(\mathcal{C}_0) = 2^{109} + \frac{1}{2}\binom{110}{55}$. Since $v_2(110!) = 105$ and $v_2(55!) = 50$, we conclude that $v_2(\binom{110}{55}) = 5$. Hence $\binom{110}{55}$ is not divisible by 64. □

Lemma 3. *If the game terminates in situation $\mathcal{C}_f$ with t numbers on the blackboard, then $g(\mathcal{C}_f)$ is divisible by 2^{t-2}.*

Proof. Let u be the largest number in $\mathcal{C}_f$, and let x be the leader with label $\ell(x)$. If $u = x$ and $\ell(x) = 1$, then every choice of coefficients for the remaining $t-1$ numbers is good, so that $g(\mathcal{C}_f) = 2t-1$. If $u = x$ and $\ell(x) = -1$, then no choice of coefficients for the remaining $t-1$ numbers is good, so that $g(\mathcal{C}_f) = 0$. If $u \neq x$, then every choice of coefficients with $c(u) = 1$ and $c(x) = \ell(x)$ is good, so that $g(\mathcal{C}_f) = 2t - 2$. □

Consider the following strategy for player B: *"Always choose a scenario $\mathcal{C}$ for which $g(\mathcal{C})$ is not divisible by 32."* Lemmas 1 and 2 ensure that this strategy can indeed be executed. Suppose for the sake of contradiction that the game would terminate in some situation $\mathcal{C}_f$ with $t \geq 7$ numbers on the blackboard. According to Lemma 3, then $g(\mathcal{C}_f)$ is a multiple of $2t - 2$ and hence divisible by 32. This contradiction implies that player B gives at most 6 cookies to player A.

Remark. If the initital situation has 1023 copies of ones on the blackboard, then the answer changes to 10 cookies. In general, if the initial situation has some odd number n of ones on the blackboard, then the answer changes to *"as many cookies as there are 1-digits in the binary representation of n"*.

Solution for Puzzle #11

The first few terms in sequence a_n are 2, 14, 198, 2.786, 39.201, 551.614. The first few terms in sequence b_n are 2, 14, 82, 478, 2.786, 16.238, 94.642, 551.614. This suggests the conjecture $a_{2k+1} = b_{3k+1}$, which can be proven in various ways. One of them reads as follows.

The recursive relation gives the three equations $a_{n+2} - 14a_{n+1} - a_n = 0$ and $a_{n+1} - 14a_n - a_{n-1} = 0$ and $a_n - 14a_{n-1} - a_{n-2} = 0$. Multiply these three equations respectively by 1, 14, -1, and add them up to get $a_{n+2} - 198a_n + a_{n-2} = 0$, or equivalently

$$a_{n+2} = 198a_n - a_{n-2}.$$

Similarly, the recursive relation gives $b_{n+3} - 6b_{n+2} + b_{n+1} = 0$ and $b_{n+2} - 6b_{n+1} + b_n = 0$ and $b_{n+1} - 6b_n + b_{n-1} = 0$ and $b_n - 6b_{n-1} + b_{n-2} = 0$ and

$b_{n-1} - 6b_{n-2} + b_{n-3} = 0$. We multiply these five equations respectively by 1, 6, 35, 6, and 1 and add them up to get $b_{n+3} - 198b_n + b_{n-3} = 0$, or equivalently

$$b_{n+3} = 198b_n - b_{n-3}.$$

We see that the two sub-sequences a_{2k+1} and b_{3k+1} have the same starting values $a_1 = b_1 = 14$ and $a_3 = b_4 = 2.786$, and that they satisfy the same linear recurrence relation. This implies the desired statement $a_{2k+1} = b_{3k+1}$ for all $k \geq 0$. Hence, there indeed exist infinitely many integers that occur in both sequences.

Solution for Puzzle #12

The answer is $n = 3$. One possible example for $n = 3$ is $x_1 = 2$ and $x_2 = x_3 = -1$, with $a_1 = 4, a_2 = 1, a_3 = 6$. For $n = 1$, the first constraint enforces $x_1 = 0$; this is in contradiction with the other two constraints. For $n = 2$, the first constraint enforces $x_2 = -x_1$. Then the second constraint is equivalent to $a_1x_1 - a_2x_1 > 0$. If we multiply this inequality by the positive value $a_1 + a_2$, we get $a_1^2x_1 - a_2^2x_1 > 0$. This is equivalent to $a_1^2x_1 + a_2^2x_2 > 0$ and contradicts the third constraint.

Solution for Puzzle #13

For $n \geq 1$, we define $c_n = n^2 - n + 2$, and we observe that

$$a_n = n^4 + 3n^2 + 4 = (n^2 - n + 2)(n^2 + n + 2) = c_n c_{n+1}.$$

For the product b_n, this implies

$$b_n = a_1 a_2 \dots a_n = c_1 c_2^2 c_3^2 \dots c_n^2 c_{n+1}.$$

Therefore, b_n will be a square whenever $c_1 c_{n+1} = 2(n^2 + n + 2)$ is a square. In other words, we are looking for solutions of the equation

$$x^2 = 2n^2 + 2n + 4,$$

which by substituting $y = 2n+1$ can be rewritten into the Pell equation

$$2x^2 = y^2 + 7.$$

This Pell equation has the fundamental solution $(x_1, y_1) = (4, 5)$. Further solutions can be constructed by the recursion $x_{k+1} = 3x_k + 2y_k$

and $y_{k+1} = 4x_k + 3y_k$. One easily sees that these solutions tend to infinity, and that in all of them y_k is odd. Hence, there exist infinitely many indices n for which b_n is a perfect square.
(An explicit answer would be that for $n = 3074$, we have $c_1 c_{n+1} = 43482$. Hence, b_{3074} is a perfect square.)

Solution for Puzzle #14

Let $f(n)$ denote the largest cardinality of a non-Balearic set in $\{1, \ldots, n\}$. One easily verifies that $f(0) = f(1) = 0$. Now consider an integer $n \geq 2$ and write it in the form $n = 2a + b$ with $0 \leq b \leq 2a - 1$. We want to show that

$$f(n) = f(2^a + b) = f(2^a - b - 1) + b.$$

Partition $\{1, 2, \ldots, n\}$ into $X = \{1, 2, \ldots, 2a - b - 1\}$ and $Y = \{2a - b, \ldots, 2a + b\}$. A non-Balearic subset S of $\{1, 2, \ldots, n\}$ contains at most $f(2a-b-1)$ elements from X (by definition of f) and at most b elements from Y (as it cannot contain $2a$ altogether, and as it contains at most one of the two numbers $2a - x$ and $2a + x$). This establishes the first inequality $f(n) \leq f(2a - b - 1) + b$.
Next, consider a non-Balearic set $T \subseteq X$ of cardinality $f(2a - b - 1)$. We claim that also $S = T \cup \{2a + 1, \ldots, 2a + b\}$ is a non-Balearic set. Suppose for the sake of contradiction that the sum $s + s'$ of some $s, s' \in S$ is a power of two. Then $s, s' \in T$ is impossible, as T itself is a non-Balearic set. Also $s, s' \in \{2a + 1, \ldots, 2a + b\}$ is impossible, as

$$2^{a+1} < (2^a + 1) + (2^a + 1) \leq s + s' \leq (2^a + b) + (2^a + b) < 2^{a+2}.$$

Hence, one of s and s' must be in T and the other one in $\{2a+1, \ldots, 2a+b\}$, which yields the final contradiction

$$2^a < s + s' \leq (2^a - b - 1) + (2^a + b) < 2^{a+1}.$$

Since the constructed non-Balearic set S is of cardinality $f(2a-b-1)+b$, we have established the second inequality $f(n) \geq f(2a - b - 1) + b$. The two established inequalities together imply the desired recursive equation $f(n) = f(2a - b - 1) + b$ displayed above.
The rest is computation.
It is easy to see (or to determine through the recursive equation) that $f(4) = 1$.
For $2^a = 8$ and $b = 3$, the recursion yields $f(11) = f(4) + 3 = 4$.

For $2^a = 32$ and $b = 20$, the recursion yields $f(52) = f(11) + 20 = 24$. For $2^a = 128$ and $b = 75$, the recursion yields $f(203) = f(52)+75 = 99$. Hence an answer to the problem is $n = 203$ with $f(203) = 99$.

(Similar computations yield $f(202) = 98$ and $f(204) = 100$. Hence, $n = 203$ constitutes the unique possible answer for the problem.)

Solution for Puzzle #15

We claim that for all $n \geq 1$ the function satisfies the bounds

$$\sqrt{n} < f(n) < \sqrt{n} + 1.$$

We prove both bounds simultaneously by induction on $n \geq 1$. For $n = 1$ with $f(1) = 3 = 2$ one indeed has $\sqrt{1} < \frac{3}{2} < \sqrt{1} + 1$. In the inductive step, we use the inductive bound $f(n) < \sqrt{n} + 1$ to get

$$f(n+1) = 1 + \frac{n}{f(n)} > 1 + \frac{n}{\sqrt{n}+1} > \sqrt{n+1}.$$

Here the inequality follows by multiplying both sides by the factor $\sqrt{n}+1$ and squaring. Furthermore, we use the inductive bound $f(n) > \sqrt{n}$ to get

$$f(n+1) = 1 + \frac{n}{f(n)} < 1 + \frac{n}{\sqrt{n}} < 1 + \sqrt{n} < 1 + \sqrt{n+1}.$$

This completes the inductive proof. Now, if we choose $m = 2018^2$, then $f(m)$ lies strictly above the lower bound $\sqrt{m} = 2018$ and strictly below the upper bound $\sqrt{m} + 1 = 2019$. Hence, $m = 2018^2$ satisfies the desired equation $\lfloor f(m) \rfloor = 2018$.

Solution for Puzzle #16

We show that $N = 5! = 120$ is the largest such integer. The lower bound construction is as follows. For every permutation of the integers $1, \ldots, 5$ create a corresponding column whose first 5 entries agree with the permutation and whose last entry (in the 6th row) equals 6.

The upper bound argument is as follows. Consider a $6 \times N$ table T with the desired properties. For each of its columns c and for every integer $x = 1, 2, \ldots, 6$ we define a new column c_x that consists of the 6 entries

$$T(1,c)+x,\ T(2,c)+x,\ T(3,c)+x,\ T(4,c)+x,\ T(5,c)+x,\ T(6,c)+x.$$

Now consider two columns i and j, and two integers x and y with $1 \leq x, y \leq 6$, and assume that the columns i_x and j_y agree componentwise modulo 6. By condition (ii) there exists a row r such that $T(r,i) = T(r,j)$. This means

$$T(r,i) + x = i_x(r) \equiv j_y(r) = T(r,j) + y = T(r,i) + y \mod 6,$$

which implies $x \equiv y \mod 6$ and hence $x = y$. If $i \neq j$, then by condition (iii) there exists a row s such that $T(s,i) \neq T(s,j)$. By using $x = y$ this then would imply the contradiction

$$\begin{aligned} T(s,i) + x = i_x(s) \equiv j_y(s) &= T(s,j) + y \\ &= T(s,j) + x \not\equiv T(s,i) + x \mod 6. \end{aligned}$$

Hence whenever two columns i_x and j_y agree componentwise modulo 6, then $i = j$ and $x = y$ must hold. This implies that the $6N$ columns c_x with $c \in T$ and $x = 1, 2, \ldots, 6$ must be pairwise distinct. By condition (i), these pairwise distinct objects correspond to pairwise distinct permutations of $1, 2, \ldots, 6$. Therefore $6N \leq 6!$, so that $N \leq 5!$.

Solution for Puzzle #17

We identify two infinite families of Aegean integers a. The first family consists of the integers of the form $a \equiv 1 \pmod 5$, as then all $n \geq 1$ satisfy

$$(a^2 + 3)a^n + 1 \equiv (1^2 + 3) \cdot 1^n + 1 \equiv 5 \equiv 0 \mod 5.$$

Consequently $a = 5b + 1$ is Aegean for $b = 1, \ldots, 403$.
The second family consists of the integers of the form $a \equiv -1 \mod 15$. Indeed if $n = 2k + 1$ is odd, then $a \equiv -1 \mod 3$ implies

$$(a^2 + 3)a^n + 1 \equiv ((-1)^2 + 3)(-1)^{2k+1} + 1 \equiv -4 + 1 \equiv 0 \mod 3.$$

On the other hand if $n = 2k$ is even, then $a \equiv -1 \mod 5$ implies

$$(a^2 + 3)a^n + 1 \equiv ((-1)^2 + 3)(-1)^{2k} + 1 \equiv 4 + 1 \equiv 0 \mod 5.$$

This yields that $a = 15c - 1$ is Aegean for $c = 1, \ldots, 134$.
Altogether, these two (disjoint) families yield at least $403 + 134 = 537$ Aegean integers in the range $\{1, 2, \ldots, 2018\}$.

Solution for Puzzle #18

Let m denote the largest number in S. We claim that for all pairs $(x_1, y_1), (x_2, y_2) \in S \times S$ with $(x_1, y_1) \neq (x_2, y_2)$ we have $x_1 + y_1 m \neq x_2 + y_2 m$. Indeed, suppose otherwise and rewrite the equation $x_1 + y_1 m = x_2 + y_2 m$ into

$$(y_1 - y_2)m = x_2 - x_1.$$

If $y_1 = y_2$, we get $x_2 - x_1 = 0$ and the contradiction $(x_1, y_1) = (x_2, y_2)$. If $y_1 \neq y_2$, we get $m = \frac{x_2 - x_1}{y_1 - y_2}$. Since $|y_1 - y_2| \geq 1$ and $|x_2 - x_1| \leq |m - 1|$, this inequality implies $m \leq m - 1$ which is ridiculous.

Consequently, the n^2 integers of the form $x + ym$ with $(x, y) \in S \times S$ are pairwise distinct.

Solution for Puzzle #19

For k, ℓ with $0 \leq k \leq r$ and $k + 1 \leq \ell \leq s$, we introduce the auxiliary value

$$T(k, \ell) := m_\ell + \sum_{i=1}^{k} m_i.$$

We claim that these $\frac{1}{2}(r+1)(2s-r)$ auxiliary values are all pairwise distinct: Let us assume that $T(k, \ell) = T(u, v)$. Without loss of generality $k \leq u$, so that this equality turns into

$$m_\ell = m_v + \sum_{i=k+1}^{u} m_i.$$

But then m_ℓ can be written as a sum of distinct other numbers in the sequence, unless $\ell = v$ and $k = u$ holds. Hence the auxiliary values indeed are distinct. As altogether there are $\frac{1}{2}(r + 1)(2s - r)$ auxiliary values, the largest value $T(r, s)$ must be at least $\frac{1}{2}(r + 1)(2s - r)$. This yields

$$\frac{1}{2}(r + 1)(2s - r) \leq T(r, s) = m_s + \sum_{i=1}^{r} m_i \leq m_s + \sum_{i=1}^{r} (m_r - r + i).$$

Here we used $m_i \leq m_r - r + i$, which follows as the sequence is increasing. The above inequality can be rewritten into

$$rs + s - r \leq r \cdot m_r + m_s,$$

which immediately implies the desired inequality from the problem statement.

Solution for Puzzle #20

This is an easy problem with many solutions. We rewrite

$$\begin{aligned} 4x^4 + y^4 - z^2 + 4xyz &= (4x^4 + y^4 + 4x^2y^2) - (4x^2y^2 + z^2 - 4xyz) \\ &= (2x^2 + y^2)^2 - (2xy - z)^2 \\ &= (2x^2 + y^2 - 2xy + z)(2x^2 + y^2 + 2xy - z). \end{aligned}$$

The two factors $A = 2x^2 + y^2 - 2xy + z$ and $B = 2x^2 + y^2 + 2xy - z$ add up to $A + B = 4x^2 + 2y^2$. If we choose the values of x and y so that $A = 4x^2 = 4 \cdot 5^{2n+2}$ and $B = 2y^2 = 2 \cdot 2^{2n}$, then the product will become $AB = 2 \cdot 10^{2n+2}$ and the sum of the digits will equal 2.
Summarizing, we pick an integer $n \geq 1$ and set $x = 5^{n+1}$ and $y = 2^n$. The desired equation $A = 4x^2$ is equivalent to $2x^2 + y^2 - 2xy + z = 4x^2$, and hence

$$z = 2x^2 + 2xy - y^2 = 2 \cdot 5^{2n+2} + 10^{n+1} - 4^n.$$

Note that z indeed is a positive integer. As the described choice of x, y, z then yields

$$4x^4 + y^4 - z^2 + 4xyz = 2 \cdot 10^{2n+2},$$

the proof is complete.

Solution for Puzzle #21

The answer is all integers $m \geq 2$. We first discuss the cases where m is a prime number.

- If m is a prime with $m \neq 3$, then any number n with $n \equiv -1 \mod m$ works: Indeed, let $n = km - 1$. Then $\gcd(m, n) = \gcd(m, km - 1) = 1$ holds and furthermore

$$\gcd(m, 4n + 1) = \gcd(m, 4km - 3) = \gcd(m, 3) = 1.$$

- If $m = 3$, then any number n with $n \equiv 1 \mod m$ works: Let $n = km + 1$. Then $\gcd(m, n) = \gcd(m, km + 1) = 1$ and $\gcd(m, 4n + 1) = \gcd(m, 4km + 5) = \gcd(m, 5) = 1$.

Finally, let $m \geq 2$ be an arbitrary integer, and let $p_1 < p_2 < \cdots < p_s$ be an enumeration of the prime factors of m. We construct a system of congruences for n with $1 \leq i \leq s$: If $p_i = 3$, then we impose $n \equiv 1 \mod p_i$. If $p_i \neq 3$, then we impose $n \equiv -1 \mod p_i$. By the Chinese Remainder Theorem, there exists an integer n that simultaneously satisfies all these congruences and that by the above discussion satisfies $\gcd(m, n) = 1$ and $\gcd(m, 4n + 1) = 1$ as desired.

Solution for Puzzle #22

Introduce the set $\mathcal{C} = \{0, 1, \ldots, m - 1\}^m$ of vectors, and define the weight $w(c) = \sum_{i=1}^m c_i m^i$ for every $c = (c_1, \ldots, c_m) \in \mathcal{C}$. Consider $c \neq c'$ in $\mathcal{C}$, and let r be the largest index with $c_r \neq c'_r$. If $c_r > c'_r$, then

$$\begin{aligned} w(c) - w(c') = \sum_{i=1}^{m} (c_i - c'_i) m^i &= (c_r - c'_r) m^r + \sum_{i=1}^{r-1} (c_i - c'_i) m^i \\ &\geq m^r - \sum_{i=1}^{r-1} (m - 1) m^i \geq m^r - (m^r - 1) = 1. \end{aligned}$$

This shows that different vectors in $\mathcal{C}$ always have different weights.

Let $A = \{a_1, \ldots, a_s\}$ be the set in the problem statement, so that every m^i (with $1 \leq i \leq m$) can be written in the form $m^i = \sum_{j=1}^s x_{i,j} a_j$ with coefficients $x_{i,1}, \ldots, x_{i,s} \in \{0, 1\}$. Then every vector weight $w(c)$ can be written in the form

$$w(c) = \sum_{i=1}^{m} c_i m^i = \sum_{i=1}^{m} c_i \sum_{j=1}^{s} x_{i,j} a_j = \sum_{j=1}^{s} \left(\sum_{i=1}^{m} c_i x_{i,j} \right) a_j.$$

Define $y_j := \sum_{i=1}^m c_i x_{i,j}$, and note that $c_i \in \{0, 1, \ldots, m - 1\}$ and $x_{i,j} \in \{0, 1\}$ imply the bounds $0 \leq y_j \leq m(m - 1)$. In other words, every vector weight can be specified by a sequence $y_1, \ldots, y_s$ of integers in the range $0 \leq y_j \leq m(m - 1)$. As there are at most m^{2s} such sequences, $|\mathcal{C}| \leq m^{2s}$ holds, which together with $|\mathcal{C}| = m^m$ implies the desired bound $|A| = s \geq \frac{m}{2}$.

Solution for Puzzle #23

We show that $N^* = 4p_1p_2p_3p_4p_5$ is the largest such integer. First we show that for the value $N = N^*$ the equation has no solution. Hence suppose for the sake of contradiction that there would exist five positive integers $x_1, \ldots, x_5$ with

$$p_1p_2p_3p_4p_5 \left(\frac{x_1}{p_1} + \frac{x_2}{p_2} + \frac{x_3}{p_3} + \frac{x_4}{p_4} + \frac{x_5}{p_5} \right) = 4p_1p_2p_3p_4p_5$$

By considering this equation modulo p_i (for $1 \le i \le 5$), we see that the integer x_i must be some multiple of p_i. But this implies $\frac{x_i}{p_i} \ge 1$, so that the value in the left hand side is at least $5p_1p_2 \ldots p_5$, a contradiction.

Next we show that for every $N > N^*$ the equation does have a solution over the positive integers. For that we prove the following generalized statement by induction on k: For every sequence $p_1 < p_2 < \cdots < p_k$ of $k \ge 1$ prime numbers and for every $N > (k-1)p_1p_2 \ldots p_k$, there exist positive integers $x_1, \ldots, x_k$ so that

$$p_1p_2 \ldots p_k \left(\frac{x_1}{p_1} + \cdots + \frac{x_k}{p_k} \right) = N \tag{1}$$

Indeed, for $k = 1$ the statement vacuously holds. In the inductive step from $k-1$ to k, we first determine x_k with $1 \le x_k \le p_k$ so that $N \equiv x_k \cdot p_1p_2 \ldots p_{k-1} \mod p_k$ holds. Then $N' \equiv (N - x_k \cdot p_1p_2 \ldots p_{k-1})/p_k$ satisfies

$$N' > \frac{1}{p_k}((k-1)p_1p_2 \ldots p_k - p_k \cdot p_1p_2 \ldots p_{k-1}) = (k-2)p_1p_2 \ldots p_{k-1}$$

By the inductive assumption, there exist positive integers $x_1, \ldots, x_{k-1}$ so that

$$p_1p_2 \ldots p_{k-1} \left(\frac{x_1}{p_1} + \cdots + \frac{x_{k-1}}{p_{k-1}} \right) = N',$$

and these $x_1, \ldots, x_{k-1}$ together with the chosen value x_k satisfy (1).

Solution for Puzzle #24

a. For two primes $p \ne q$, we choose $r := p + q - 1$. Then $pq - r = (1-p)(1-q)$, and $p - r = 1 - q$, and $q - r = 1 - p$. As $pq - r$ is divisible by $p - r$ and by $q - r$ (and as there are infinitely many primes), these three integers form a Somerset-triple (p, q, r).

b. Let us consider a Somerset-triple (p, q, r) with $\max\{p, q\} \geq 2r$; by symmetry we may assume $p \geq q$, which then yields $p > 2r$. Since $p - r$ divides $pq - r$, it also divides

$$(pq - r) - (p - r)q = r(q - 1).$$

Since p is a prime with $p > r$, we have $1 = \gcd(p, r) = \gcd(p-r, r)$. Therefore $p-r$ must divide the other factor $q-1$, so that $(p-r)k = q - 1$ for some integer $k \geq 1$. Now $p \geq q$ and $p > 2r$ yield

$$p > q - 1 = (p - r)k > \frac{p}{2},$$

which implies $2 > k$. Hence $k = 1$ and $p - r = q - 1$. Since $q - r$ divides $pq - r$, it also divides

$$(pq - r) - (q - r)(p + r) = r(r + p - q - 1) = 2r(r - 1)$$

This implies $q - r \leq 2r(r - 1)$, and hence $q \leq 2r^2 - r$. From $p - r = q - 1$ we get $p = q + r - 1 \leq 2r^2 - 1$. The discussion implies that all Somerset-triples (p, q, r) satisfy $\max\{p, q\} \leq 2r^2$. Hence there are only finitely many Somerset-triples (p, q, r) with $r \leq 1.000.000$.

Remark. The upper bound $\max\{p, q\} \leq 2r^2$ can be improved to $\max\{p, q\} \leq 4r - 3$, and no further.

Solution for Puzzle #25

We will show that there exist infinitely many such pairs. We first observe that $a^2 - a + b^3 + 1$ is a multiple of ab if and only if

$$a^2 - a + 1 \equiv 0 \mod b \quad \text{and} \quad b^3 + 1 \equiv 0 \mod a. \tag{2}$$

We next prove that whenever (a, b) is a solution, then also the pair (x, y) with $x = \frac{b^3+1}{a}$ and $y = \frac{x^2-x+1}{b}$ is a solution.

(i) Since $b^3 + 1 \equiv 0 \mod a$ by (2), we get that $x = \frac{b^3+1}{a}$ is integer.

(ii) From $xa = b^3 + 1$ we conclude $xa \equiv 1 \mod b$ and $\gcd(a, b) = 1$.

(iii) Now $a^2(x^2-x+1) \equiv (ax)^2-(ax)a+a^2 \equiv a^2-a+1 \equiv 0 \mod b$. Since b divides $a^2(x^2-x+1)$ and since $\gcd(a,b)=1$ by (ii), we get that $y = \frac{x^2-x+1}{b}$ is integer.

(iv) From $yb = x^2-x+1$ we get that $x^2-x+1 \equiv 0 \mod y)$.

(v) From $yb = x^2-x+1$ we get $yb \equiv 1 \mod x$, and from $xa = b^3+1$ we get $b^3+1 \equiv 0 \mod x)$.

(vi) Now (v) yields $y^3+1 \equiv y^3+(yb)^3 \equiv y^3(b^3+1) \equiv 0 \mod x$.

Summarizing, (i) and (iii) yield that x and y are indeed integers, and (iv) and (vi) yield that they satisfy the desired divisibility conditions $x^2-x+1 \equiv 0 \mod y$ and $y^3+1 \equiv 0 \mod x$.

Finally, we use the above observation to construct an infinite sequence of solutions. We start with the trivial solution $(a_0, b_0) = (1,1)$, and for $i \geq 1$ we define $a_{i+1} = \frac{b_i^3+1}{a_i}$ and $b_{i+1} = \frac{a_{i+1}^2-a_{i+1}+1}{b_i}$. The first few pairs in the solution sequence are $(1,1)$ and $(2,3)$ and $(14,61)$ and $(16.213, 4.308.937)$.

We prove by induction on $i \geq 1$ that $a_i < b_i < a_{i+1}$. These inequalities indeed hold for $i=1$. In the inductive step from $i-1$ to i, we use the inductive assumption $b_{i-1} < a_i$ to derive

$$b_i = \frac{a_i^2-a_i+1}{b_{i-1}} \geq \frac{a_i^2-a_i+1}{a_i-1} > a_i$$

and we use the just the just derived inequality $b_i > a_i$ to get

$$a_{i+1} = \frac{b_i^3+1}{a_i} > \frac{b_i^3+1}{b_i} > b_i$$

Since $a_1 < a_2 < a_3 < \ldots$, all the constructed solutions are pairwise distinct.

Solution for Puzzle #26

Notations and preliminaries If there exists an axes-parallel cube that has all points of T in its interior or on its boundary and all points of $S \setminus T$ in its exterior, we say that this cube *separates* T from $S \setminus T$. For an axes-parallel cube C, we denote by C_x, C_y, C_z its projections on the three axes; note that C_x, C_y, C_z are three closed intervals that all have the same length. A point p with coordinates (x_p, y_p, z_p) is contained in cube C, if and only if simultaneously $x_p \in C_x$ and $y_p \in C_y$ and $z_p \in C_z$.

The upper bound argument Suppose that there exists a 6-element set $S = \{p_1, \ldots, p_6\}$ with the desired property. For $i = 1, \ldots, 6$, let (x_i, y_i, z_i) denote the coordinates of point p_i. Let p_a (p_b) be a point with largest (smallest) x-coordinate, let p_c (p_d) be a point with largest (smallest) y-coordinate, and let p_e (p_f) be a point with largest (smallest) z-coordinate.

We claim that the six indices a, b, c, d, e, f are pairwise distinct: Otherwise, let p_k be a point with $k \notin \{a, b, c, d, e, f\}$, let $T = S \setminus \{p_k\}$, and let C be an axes-parallel cube that separates T from $\{p_k\}$. As the interval C_x contains the extreme points x_a and x_b, it does also contain x_k; analogously we get that C_y contains y_k and that C_z contains z_k. This yields the contradiction $p_k \in C$.

Hence $\{a, b, c, d, e, f\} = \{1, 2, 3, 4, 5, 6\}$. By symmetry we may assume that $x_a - x_b \geq y_c - y_d \geq z_e - z_f$. Consider the cube C that separates $T = \{p_a, p_b, p_c, p_d\}$ from the remaining two points $\{p_e, p_f\}$.

- Since the interval C_x contains x_a as well as x_b, it contains all six x-coordinates. Furthermore, the sidelength ℓ of C is at least $x_a - x_b$.
- Since the interval C_y contains y_c as well as y_d, it contains all six y-coordinates.
- Since the cube C does contain neither point p_e nor point p_f, the interval C_z (with length $\ell \geq x_a - x_b$) must be strictly contained in the interval $[z_f, z_e]$ (with length $z_e - z_f \leq x_a - x_b \leq \ell$). That's impossible.

This contradiction shows $n \leq 5$.

The lower bound argument Consider the three points $p_1 = (2, 2, 0)$, $p_2 = (-2, 1, 0)$, $p_3 = (1, -2, 0)$ in the xy-plane. It is straightforward to see that for each of the eight subsets $U \subseteq \{p_1, p_2, p_3\}$, there exists an axes-parallel quadrant that contains U together with the origin $(0, 0, 0)$ but does not contain $\{p_1, p_2, p_3\} \setminus U$. We list these eight quadrants for completeness:

- For $U = \varnothing$, use the quadrant $x \leq 0$ and $y \leq 0$.
- For $U = \{p_1\}$, use the quadrant $x \geq 0$ and $y \geq 0$.
- For $U = \{p_2\}$, use the quadrant $x \leq 0$ and $y \geq 0$.
- For $U = \{p_3\}$, use the quadrant $x \geq 0$ and $y \leq 0$.
- For $U = \{p_1, p_2\}$, use the quadrant $x \leq 2$ and $y \geq 0$.
- For $U = \{p_1, p_3\}$, use the quadrant $x \geq 0$ and $y \leq 2$.
- For $U = \{p_2, p_3\}$, use the quadrant $x \leq 1$ and $y \leq 1$.
- For $U = \{p_1, p_2, p_3\}$, use the quadrant $x \leq 2$ and $y \leq 2$.

Now add the two points $p_4 = (0, 0, 100)$ and $p_5 = (0, 0, -100)$ to the picture. Consider a subset $T \subseteq \{p_1, p_2, p_3, p_4, p_5\}$, and let $U = T \cap \{p_1, p_2, p_3\}$ and let $V = T \cap \{p_4, p_5\}$. There exists a separating cube for T whose projection on the xy-plane belongs to the quadrant for set U, such that the projection of one vertical edge coincides with the vertex of this quadrant.

- For $V = \varnothing$, top and bottom face of the cube lie in the planes $z = 10$ and $z = -10$.
- For $V = \{p_4\}$, top and bottom face of the cube lie in the planes $z = 0$ and $z = 100$,
- For $V = \{p_5\}$, top and bottom face of the cube lie in the planes $z = 0$ and $z = -100$,
- For $V = \{p_4, p_5\}$, top and bottom face of the cube lie in the planes $z = 100$ and $z = -100$,

Summarizing, we see that the set $\{p_1, p_2, p_3, p_4, p_5\}$ has all desired properties. This shows $n \geq 5$.